Orthogonal matrices

01. Since the matrix is 3×3, to find the third column, it is sufficient to calculate the cross product of the two columns of the matrix, i.e. $(1/\sqrt{2}, -1/\sqrt{2}, 0) \wedge (1/\sqrt{6}, 1/\sqrt{6}, 2/\sqrt{6})$. The result is $(-2/\sqrt{12}, -2/\sqrt{12}, 2/\sqrt{12})$ or, to simplify, $(-1/\sqrt{3} - 1/\sqrt{3}, 1/\sqrt{3})$, so the matrix is
$$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$$
But the third column can also be the opposite of the cross product, so the other possible matrix is:
$$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \end{pmatrix}$$

02. Since Q must be orthogonal, the module of the first column must be 1. Then to find the entry q_{31} we must set $(-2/3)^2 + (2/3)^2 + q_{31}^2 = 1$. We find $q_{31} = \pm 1/3$.
The same argument applies to the second column and we find $q_{32} = \pm 2/3$.
But the two columns must be orthogonal so, if $q_{31} = 1/3$ then $q_{32} = 2/3$ and if $q_{31} = -1/3$ then $q_{32} = -2/3$
Since the matrix is 3×3, to find the third column, it is sufficient to calculate the cross product of the two columns of the matrix.
In the first case $(-2/3, 2/3, 1/3) \wedge (2/3, 1/3, 2/3) = (1/3, 2/3, -2/3)$
In the second case $(-2/3, 2/3, -1/3) \wedge (2/3, 1/3, -2/3) = (-1/3, -2/3, -2/3)$
But the third column can also be the opposite of the cross product, so there are four possible ways to construct the matrix:
$$\begin{pmatrix} -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \\ 1/3 & 2/3 & -2/3 \end{pmatrix} \begin{pmatrix} -2/3 & 2/3 & -1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} -2/3 & 2/3 & -1/3 \\ 2/3 & 1/3 & -2/3 \\ -1/3 & -2/3 & -2/3 \end{pmatrix} \begin{pmatrix} -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \\ -1/3 & -2/3 & 2/3 \end{pmatrix}$$

Spaces with a scalar product

11. a. Since the space is very simple, we can avoid Gram-Schmidt process.
First we calculate an orthogonal basis for W. First vector is $(1, 1, 0)$.
Second vector is a vector of W, i.e. a linear combination of the two generators of W:
$a(1, 1, 0) + b(0, 1, 1) = (a, a + b, b)$. This vector should be orthogonal to $(0, 1, 1)$, that is, we must have: $\langle (a, a + b, b), (0, 1, 1)\rangle = 0 \quad \Rightarrow \quad 2a + b = 0$.
By instance $a = 1$; $b = -2$, hence the vector $(1, -1, -2)$.
By normalization we get the o.n. basis $\dfrac{(1, 1, 0)}{\sqrt{2}}, \dfrac{(1, -1, -2)}{\sqrt{6}}$.
The projection of v onto W must be calculated by means of the o.n. basis and is
$$p = \left\langle \frac{(1, 1, 0)}{\sqrt{2}}, (1, 2, 0)\right\rangle \frac{(1, 1, 0)}{\sqrt{2}} + \left\langle \frac{(1, -1, -2)}{\sqrt{6}}, (1, 2, 0)\right\rangle \frac{(1, -1, -2)}{\sqrt{6}} =$$
$$= \frac{3}{2}(1, 1, 0) - \frac{1}{6}(1, -1, -2) = \left(\frac{4}{3}, \frac{5}{3}, \frac{1}{3}\right).$$
b. We have $p - v = \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)$. This vector is orthogonal to $(1, 1, 0)$ and to $(0, 1, 1)$ and so, by bilinearity it is orthogonal to any linear combination or these two vectors, i.e. to any vector in W.

12. To find the distance, we must calculate the projection of the vector $(1, 1, 1, 1)$ onto W. To do this we must find an orthonormal basis of W by means of Gram-Schmidt algorithm.
Set $v_1 = (0, 1, 0, 0), v_2 = (0, 0, 1, 2), v_3 = (1, 1, 1, 0)$. Then:
$v_1'' = (0, 1, 0, 0)$ (since its module is 1)
$v_2'' = (0, 0, 1, 2)/\sqrt{5}$ (v_2 is already orthogonal to v_1. It is sufficient to normalize it)
$v_3' = (1, 1, 1, 0) - \langle (1, 1, 1, 0), (0, 1, 0, 0)\rangle (0, 1, 0, 0) - \langle (1, 1, 1, 0), (0, 0, 1, 2)/\sqrt{5}\rangle (0, 0, 1, 2)/\sqrt{5} =$
$= (1, 1, 1, 0) - (0, 1, 0, 0) - (0, 0, 1/5, 2/5) = (1, 0, 4/5, -2/5)$.
To simplify calculation, set $v_3' = (5, 0, 4, -2)$, and get v_3'' by normalization: $v_3'' = (5, 0, 4, -2)/\sqrt{45}$
Now the projection is $\langle (1, 1, 1, 1), (0, 1, 0, 0)\rangle (0, 1, 0, 0) + \langle (1, 1, 1, 1), (0, 0, 1, 2)/\sqrt{5}\rangle (0, 0, 1, 2)/\sqrt{5} +$
$\langle (1, 1, 1, 1), (5, 0, 4, -2)/\sqrt{45}\rangle (5, 0, 4, -2)/\sqrt{45} = (0, 1, 0, 0) + 3/5(0, 0, 1, 2) + 7/45(5, 0, 4, -2)$
Final result is $p = (7/9\ ,\ 1\ ,\ 11/9\ ,\ 8/9)$.

In order to be sure that there is no calculation error, it is advisable to verify that $p - v = $ $= (7/9\ ,\ 1\ ,\ 11/9\ ,\ 8/9) - (1,1,1,1) = (2/9\ ,\ 0\ ,\ -2/9\ ,\ 1/9)$ is orthogonal to W, i.e. to each of the three vectors v_1, v_2, v_3. We omit this standard calculation.
The distance is the module of the vector $p - v = (2/9\ ,\ 0\ ,\ -2/9\ ,\ 1/9)$, that is $\sqrt{5/81}$.

13. The matrix is definite positive. This can be verified in several ways, by instance by observing that the two principal minors of $A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$ are positive. This is enough to prove that it induces the following scalar product: $\langle (x,y),(x_1,y_1)\rangle_* = (x \quad y) \cdot A \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$
To find an an o.n. basis of $\mathbb{R}^2$ we apply Gram-Schmidt process to the basis $v_1(1,0), v_2(0,1)$:
First normalize v_1: $\| v_1 \|_* = \sqrt{\langle (1,0),(1,0)\rangle_*} = \sqrt{(1\ 0)\begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \sqrt{2} \Rightarrow v_1'' = \dfrac{(1,0)}{\sqrt{2}}$
Then $v_2' = (0,1) - \left\langle (0,1), \dfrac{(1,0)}{\sqrt{2}} \right\rangle_* \dfrac{(1,0)}{\sqrt{2}} = (0,1) - \langle (0,1),(1,0)\rangle_* \dfrac{(1,0)}{2}$.
We must calculate the scalar product: $\langle (0,1),(1,0)\rangle_* = (0\ 1)\begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2$
So $v_2' = (0,1) - 2\,\dfrac{(1,0)}{2} = (-1,1)$. To conclude the process, we must normalize v_2'
$\| v_2' \| = \sqrt{\langle (-1,1),(-1,1)\rangle_*} = \sqrt{(-1\ 1)\begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}} = \sqrt{3} \quad \Rightarrow \quad v_2'' = \dfrac{(-1,1)}{\sqrt{3}}$

14. a. The product is a scalar product only if the matrix is definite positive. We can easily check it by using Sylvester's law of inertia and reducing A both by rows and columns:
$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 8 & k \\ 0 & k & 1 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ C2 \to C_2 - 2C_1 \end{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & k \\ 0 & k & 1 \end{pmatrix} \begin{matrix} R_3 \to R_3 - (k/4)R_2 \\ C3 \to C_3 - (k/4)C_2 \end{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 - k^2/4 \end{pmatrix}$$
By Sylvester's law of inertia, the eigenvalues of the reduced matrix have the same signs as those of the matrix A, so they are all positive if and only if $1 - k^2/4 > 0$ that is if and only if $-2 < k < 2$.

b. Let us calculate all the three scalar products
$$\langle (1,0,1),(1,1,0)\rangle_* = (1 \quad 0 \quad 1)\begin{pmatrix} 1 & 2 & 0 \\ 2 & 8 & k \\ 0 & k & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = (1 \quad 2+k \quad 1)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 3 + k$$
So they are orthogonal if $k = -3$, but for this k, the product $\langle\, ,\rangle_*$ is not a scalar product.
In the same way, we have:
$\langle (1,0,1),(0,1,0)\rangle_* = 2 + k$ but if $k = -2$, the product $\langle\, ,\rangle_*$ is not a scalar product.
$\langle (1,0,1),(-1,1,0)\rangle_* = 1 + k$. This time, if $k = -1$, the product $\langle\, ,\rangle_*$ is a scalar product and the two vectors are orthogonal.

c. Let $k = -1$. We must choose a basis of $\mathbb{R}^3$ and apply Gram-Schmidt algorithm. Since the two vectors $(1,0,1), (-1,1,0)$ are already orthogonal, it is convenient to choose a basis that contains the two vectors, by instance $\mathcal{B} : v_1(1,0,1), v_2(-1,1,0), v_3(0,0,1)$. This way the first two step of the algorithm are very simple:
First normalize v_1: We have $\| v_1 \|^2 = (1\ 0\ 1) \cdot A \cdot (1\ 0\ 1)^T = 2$. Hence $v_1'' = (1,0,1)/\sqrt{2}$.
Then normalize v_2: We have $\| v_2 \|^2 = (-1\ 1\ 0) \cdot A \cdot (-1\ 1\ 0)^T = 5$. Hence $v_2'' = (-1,1,0)/\sqrt{5}$.
$v_3' = (0,0,1) - \langle (0,0,1),(1,0,1)/\sqrt{2}\rangle_* (1,0,1)/\sqrt{2} - \langle (0,0,1),(-1,1,0)/\sqrt{5}\rangle_* (-1,1,0)/\sqrt{5} =$
We must calculate the two scalar products:
$\langle (0,0,1),(1,0,1)/\sqrt{2}\rangle_* = (0\ 0\ 1) \cdot A \cdot (1\ 0\ 1)^T = 1/\sqrt{2}$
$\langle (0,0,1),(-1,1,0)/\sqrt{5}\rangle_* = (0\ 0\ 1) \cdot A \cdot (-1\ 1\ 0)^T/\sqrt{5} = 1/\sqrt{5}$. So $v_3' = (-7/10\ ,\ 1/5\ ,\ 1/2)$.
To simplify calculation set $v_3'(-7,2,5)$, and get v_3'' by normalization:
$\| v_3'' \|^2 = (-7,2,5) \cdot A \cdot (-7,2,5)^T = 30$.
Finally, we get the following o.n. basis: $(1,0,1)/\sqrt{2} \quad , \quad (-1,1,0)/\sqrt{5} \quad , \quad (-7,2,5)/\sqrt{30}$.

15. The inequality is $|\langle f_1, f_2\rangle| \leq \| f_1 \|\| f_2 \|$. So we must calculate three scalar products:
$\langle f_1, f_2\rangle = \int_1^2 \frac{ax^2+b}{x}\cdot\frac{1}{x}\,dx = \int_1^2 \frac{ax^2+b}{x^2}\,dx = \left[ax - \frac{b}{x}\right]_1^2 = a + \frac{b}{2}$
$\| f_1 \|^2 = \langle f_1, f_1\rangle = \int_1^2 \left(\frac{ax^2+b}{x}\right)^2 dx = \left[\frac{a^2x^3}{3} + 2abx - \frac{b^2}{x}\right]_1^2 = \frac{7}{3}a^2 + 2ab + \frac{b^2}{2}$
$\| f_2 \|^2 = \langle f_2, f_2\rangle = \int_1^2 \left(\frac{1}{x}\right)^2 dx = \left[-\frac{1}{x}\right]_1^2 = \frac{1}{2}$
So we must verify that $\left|a + \frac{b}{2}\right| \leq \sqrt{\left(\frac{7}{3}a^2 + 2ab + \frac{b^2}{2}\right)\frac{1}{2}}$.
Taking squares: $a^2 + \frac{b^2}{4} + ab \leq \frac{7}{6}a^2 + ab + \frac{b^2}{4}$ which is obviously true for any a and b.
Furthermore it is an equality when $a = 0$ and any b.

16. Set $v_1 = 1, v_2 = x, v_3 = x^2$.

a. To find an an orthonormal basis of V we apply Gram-Schmidt process to the basis v_1, v_2:
First normalize v_1. But $\| v_1 \|^2 = \int_0^1 1\,dx = 1$. Hence $v_1'' = 1$.
Then $v_2' = x - \langle 1, x\rangle\, 1 = x - \left(\int_0^1 x\,dx\right) 1 = x - \frac{1}{2}$. To conclude we must normalize v_2'
$\| v_2' \|^2 = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right]_0^1 = \frac{1}{12}$. Hence $v_2'' = \sqrt{12}\left(x - \frac{1}{2}\right) = \sqrt{3}(2x-1)$
To find an an orthonormal basis of V_1 we must continue the Gram-Schmidt process up to the vector v_3:
$v_3' = x^2 - \langle 1, x^2\rangle\, 1 - \langle \sqrt{3}(2x-1), x^2\rangle\, \sqrt{3}(2x-1) =$
$= x^2 - \left(\int_0^1 x^d x\right) 1 - 3\left(\int_0^1 (2x^3 - x^2)dx\right)(2x-1) = x^2 - \frac{1}{3} - 3\left[\frac{x^4}{2} - \frac{x^3}{3}\right]_0^1 (2x-1) =$
$= x^2 - \frac{1}{3} - \frac{1}{2}(2x-1) = x^2 - x + \frac{1}{6}$.
To conclude we must normalize v_3'
$\| v_3' \|^2 = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx = \left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{4x^3}{9} - \frac{x^2}{6} + \frac{1}{36}x\right]_0^1 = \frac{1}{180}$
So $v_3'' = \left(x^2 - x + \frac{1}{6}\right)\sqrt{180} = \sqrt{5}(6x^2 - 6x + 1)$

b. The projection of f onto V must be calculated by means of the o.n. basis and is
$p = \langle 1, x^2\rangle\, 1 + \langle\sqrt{3}(2x-1), x^2\rangle\sqrt{3}(2x-1) = \frac{1}{3}\cdot 1 + 3\frac{1}{6}(2x-1) = x - \frac{1}{6}$
The relation between $f(x)$ and its projection $p(x)$ is that $f(x) - p(x)$ is orthogonal to V. In fact $f(x) - p(x) = x^2 - x + \frac{1}{6}$ and, as it follows from the previous calculations, this function is orthogonal to 1 and to $\sqrt{3}(2x-1)$ and so, by bilinearity it is orthogonal to any linear combination or these two functions, i.e. to any function in V.
The minimum property means that the distance between f and p is less or equal to the distance between p and any function in V.
The distance is $\| f(x) - p(x) \| = \left\| x^2 - x + \frac{1}{6} \right\| = \frac{1}{\sqrt{180}}$, as already calculated.

17. Put $v_1 = 1/x, v_2 = x$.

a. To find an an orthonormal basis of V we apply Gram-Schmidt process to the basis v_1, v_2:
First normalize v_1. But $\| v_1 \|^2 = \int_1^2 \frac{1}{x^2}\,dx = \frac{1}{2}$. Hence $v_1'' = \frac{\sqrt{2}}{x}$.

Then $v_2' = x - \left\langle \frac{\sqrt{2}}{x}, x \right\rangle \frac{\sqrt{2}}{x} = x - 2\left(\int_1^2 dx\right)\frac{1}{x} = x - \frac{2}{x}$. Now we must normalize v_2'

$\| v_2' \|^2 = \int_0^1 \left(x - \frac{2}{x}\right)^2 dx = \left[\frac{x^3}{3} - \frac{4}{x} - 4x\right]_1^2 = \frac{1}{3}$. Hence $v_2'' = \sqrt{3}\left(x - \frac{2}{x}\right)$

b. The projection of f onto V must be calculated by means of the o.n. basis and is

$$p = \left\langle x^2, \frac{\sqrt{2}}{x} \right\rangle \frac{\sqrt{2}}{x} + \left\langle x^2, \sqrt{3}\left(x - \frac{2}{x}\right)\right\rangle \sqrt{3}\left(x - \frac{2}{x}\right) = 2\left(\int_1^2 x\,dx\right)\frac{1}{x} +$$

$$+3\left(\int_1^2 (x^3 - 2x)\,dx\right)\left(x - \frac{2}{x}\right) = 2\left(\frac{3}{2}\right)\frac{1}{x} + 3\left(\frac{3}{4}\right)\left(x - \frac{2}{x}\right) = -\frac{3}{2x} + \frac{9}{4}x$$

c. We have: $\int_1^2 f(x)p(x)\,dx = \langle f(x), p(x)\rangle = \left\langle f(x), \left\langle f(x), \frac{1}{x}\right\rangle \frac{1}{x} + \langle f(x), x\rangle x \right\rangle$

By bilinearity, the second side is

$$\left\langle f(x), \frac{1}{x}\right\rangle \left\langle f(x), \frac{1}{x}\right\rangle + \langle f(x), x\rangle \langle f(x), x\rangle = \left\langle f(x), \frac{1}{x}\right\rangle^2 + \langle f(x), x\rangle^2 =$$

$= \left(\int_1^2 f(x)\frac{1}{x}\,dx\right)^2 + \left(\int_1^2 f(x)x\,dx\right)^2$ From here easily the conclusion.

18. a. We have to prove that $\langle , \rangle_1$ is symmetric, bilinear and positive. The first two are obvious. As for the third one, observe that $\langle f, f\rangle_1 = \int_{-1}^1 f^2(x)x^2\,dx$ cannot be negative since $f^2(x)x^2$ is non negative and $-1 < 1$. The product $\langle f, f\rangle_1$ can be 0 if and only if f is the null function since x^2 has only one zero in $[-1, 1]$.

b. Just apply Gram-Schmidt process; let us set $v_1 = 1$ and $v_2 = x$

First normalize v_1. But $\| v_1 \|_1^2 = \int_{-1}^1 1 \cdot 1 \cdot x^2\,dx = \frac{2}{3}$. Hence $v_1'' = \sqrt{\frac{3}{2}}$.

Now observe that $\langle 1, x\rangle_1 = 0$ since $\int_{-1}^1 1 \cdot x \cdot x^2\,dx = 0$. This means that to conclude we have only to normalize v_2. But $\| v_2 \|_1^2 = \int_{-1}^1 x \cdot x \cdot x^2\,dx = \frac{2}{5}$. Hence $v_2'' = \sqrt{\frac{5}{2}}\,x$.

19. Let us calculate $\langle 2 - x, x\rangle = \int_0^a (2 - x)x\,dx = \int_0^a 2x - x^2\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^a = a^2 - \frac{a^3}{3}$

So $\langle 2 - x, x\rangle = 0$ only if $a = 3$ (we exclude $a = 0$).

To calculate the projection we must build an orthonormal basis for the subspace. Since the two functions are orthogonal, we only need to normalize them.

$\| 2 - x \|^2 = \int_0^3 (2 - x)^2\,dx = \int_0^3 4 - 4x + x^2\,dx = \left[4x - 2x^2 + \frac{1}{3}x^3\right]_0^2 = 3$

$\| x \|^2 = \int_0^3 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = 9$ The o.n. basis is: $\frac{2 - x}{\sqrt{3}}$; $\frac{x}{3}$

The projection is:

$$p = \left\langle x^2, \frac{2 - x}{\sqrt{3}}\right\rangle \frac{2 - x}{\sqrt{3}} + \left\langle x^2, \frac{x}{3}\right\rangle \frac{x}{3} = \left(\frac{1}{3}\int_0^3 2x^2 - x^3\,dx\right)(2 - x) + \left(\frac{1}{9}\int_0^3 x^3\,dx\right)x =$$

$$= \frac{1}{3}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^3 (2 - x) + \frac{1}{9}\left[\frac{x^4}{4}\right]_0^3 x = \frac{1}{3}\left(-\frac{9}{4}\right)(2 - x) + \frac{1}{9}\frac{81}{4}x = -\frac{3}{2} + 3x$$

Matrix norms and condition number

21. Since A is symmetric it suffices to calculate its eigenvalues: $\det\begin{pmatrix} 1 - x & -2 \\ -1 & -1 - x \end{pmatrix} = x^2 - 5$

So $\lambda_1 = \sqrt{5}$ $\lambda_2 = -\sqrt{5}$. It follows that $\| A \|_2 = \sqrt{5}$ and $\text{cond}_2(A) = 1$.

22. Since A is non-symmetric we must use A^TA

$$A^TA = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

The eigenvalues of A^TA are $\lambda_1 = 2 \quad \lambda_2 = 8 \quad \lambda_3 = 9$, so its singular values are $\sqrt{2}$, $2\sqrt{2}$, 3. It follows that $\| A \|_2 = 3$ and $\text{cond}_2(A) = 3/\sqrt{2}$.
The 1-norms of the columns of A are all 3, so $\| A \|_1 = 3$.
The 1-norms of the rows of A are 2, 4, 3, so $\| A \|_\infty = 4$.
To calculate $\text{cond}_1(A)$ and $\text{cond}_\infty(A)$ we need to use A^{-1}.

With few passages we get $A^{-1} = \begin{pmatrix} 1/2 & 1/4 & 0 \\ 1/2 & -1/4 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$

The 1-norms of the columns of A^{-1} are 1, 1/2, 1/3, so $\| A^{-1} \|_1 = 1$.
The 1-norms of the rows of A^{-1} are 3/4, 3/4, 1/3, so $\| A^{-1} \|_\infty = 3/4$.
We conclude: $\text{cond}_1(A) = \| A \|_1 \cdot \| A^{-1} \|_1 = 3$ and $\text{cond}_\infty(A) = \| A \|_\infty \cdot \| A^{-1} \|_\infty = 3$

23. a. Since A is symmetric, we get $\| A \|_2 = 5$.
The 1-norms of the columns of A are $5, 7, 5$, so $\| A \|_1 = 7$
Since A is symmetric, we have $\| A \|_\infty = \| A \|_1 = 7$.

b. From the given data we have $\text{cond}_2(A) = 5/2$

c. We have $\| b \| = \sqrt{3}$ and $\| \delta b \| = \| b - b_1 \| = 1$.
The well-known inequality is $\dfrac{\| x - x_1 \|_2}{\| x \|_2} \le \text{cond}_2(A) \dfrac{\| \delta b \|}{\| b \|} = \dfrac{5}{2}\dfrac{1}{\sqrt{3}} = \dfrac{5}{2\sqrt{3}} \simeq 1.44$

24. a. We must calculate $\det(A - xI)$.
It is advantageous to make some elementary operations on the matrix $A - xI$

$$\det \begin{pmatrix} -1-x & 2 & -1 \\ 2 & 2-x & -2 \\ -1 & -2 & -1-x \end{pmatrix} = \overset{[R_3 \to R_3 + R_1]}{\det} \begin{pmatrix} -1-x & 2 & -1 \\ 2 & 2-x & -2 \\ -2-x & 0 & -2-x \end{pmatrix} =$$

$$= \overset{[C_3 \to C_3 - C_1]}{\det} \begin{pmatrix} -1-x & 2 & x \\ 2 & 2-x & -4 \\ -2-x & 0 & 0 \end{pmatrix} = (-2-x)\det \begin{pmatrix} 2 & x \\ 2-x & -4 \end{pmatrix} =$$

$= (-2-x)(-8-2x+x^2)$
Since the roots of the quadratic polynomial $-8-2x+x^2$ are 4 and -2, we deduce that the eigenvalues of A are $-2, -2, 4$.

b. From the given data we have $\| A \|_2 = 4$ and $\text{cond}_2(A) = 4/2 = 2$

c. We remark that if λ is an eigenvalue of A then $\lambda + k$ is an eigenvalue of $A + kI$. So:
The eigenvalues of $A + I$ are $-1, -1, 5$ and $\text{cond}_2(A + I)$=5
The eigenvalues of $A - I$ are $-3, -3, 3$ and $\text{cond}_2(A - I)$=1
The eigenvalues of $A - 2I$ are $-4, -4, 2$ and $\text{cond}_2(A - 2I)$=2
$A - I$ has the best condition number and $A + I$ has the worst one.

25. From the given data we can calculate:

$\text{cond}_2(A) \simeq \sqrt{\dfrac{91.6986}{0.0022}} = \sqrt{41681.182} \simeq 204.160$ In our case:

$b = (1, 4, 4, 4)$ and $\| b \| = 7 \qquad \delta b = (0.1, 0, -0.1, 0)$ and $\| \delta b \| = 0.02 \qquad \| x \| \simeq 2.42$

The well-known inequality can be written as

$\| \delta x \| \le \| x \| \ \text{cond}(A) \dfrac{\| \delta b \|}{\| b \|} \simeq 2.42 \cdot 204 \dfrac{0.02}{7} \simeq 1.411$

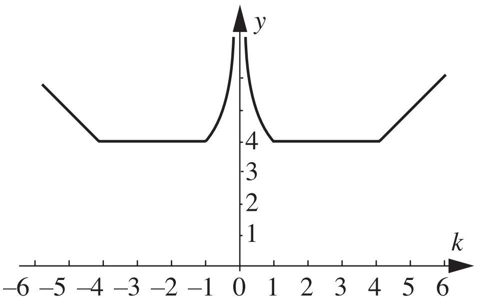
From here, without any further calculation, we can only say that the distance of each component of x_1 from the corresponding component of x cannot be bigger than 1.411. By example, if $x = (a, b, c, d)$, then $a = 2/3 \pm 1.411$.

26. Since A is non-symmetric, me must consider $A_k^T \cdot A_k = \begin{pmatrix} 4 & 6 & 0 \\ 6 & 13 & 0 \\ 0 & 0 & k^2 \end{pmatrix}$

    Its eigenvalues are $1, 16, k^2$ and its singular values are $1, 4, \mid k \mid$.
    This means that we can can calculate $\text{cond}_2(A_k)$ for all $k \neq 0$. There are three cases:

    If $1 \leq\mid k \mid\leq 4$ then $\text{cond}_2(A_k) = 4$
    If $\mid k \mid< 1$ then $\text{cond}_2(A_k) = 4/\mid k \mid$
    If $\mid k \mid> 4$ then $\text{cond}_2(A_k) =\mid k \mid /1$

    From here one can easily draw the graphic of the function $y = f(k)$ which is symmetric with respect to the y axis.



27. a. Since A must be orthogonal, the module of the first column must be 1. Then to find the entry a_{31} we must set $(6/7)^2 + (2/7)^2 + a_{31}^2 = 1$. We find $a_{31} = \pm 3/7$. Let us choose $a_{31} = 3/7$. Now, by symmetry $a_{12} = 2/7$ and $a_{13} = 3/7$.
    Since first column and second column should be orthogonal, then the entry a_{32} must be $-6/7$ and $a_{32} = a_{23}$. Finally $a_{33} = 2/7$, since third column is orthogonal to the other ones.
    The matrix A is orthogonal, so $\text{cond}_2(A) = 1$.
    Since A is symmetric, its eigenvalues are all real and can only be 1 and -1. It is impossible for all the eigenvalues to be 1, because in this case A would be I and by the same argument all the eigenvalues cannot be -1.

    b. The eigenvalues of $A - kI$ are $1 - k$ and $-1 - k$, so it makes sense to calculate $\text{cond}_2(A - kI)$ for $k \neq \pm 1$.
    One can easily check that:
    If $k > 1$, then the absolute values of the eigenvalues are $k + 1, k - 1$ and $k + 1 > k - 1$, so in this case $\text{cond}_2(A) = \dfrac{k+1}{k-1}$
    If $0 < k < 1$, then the absolute values of the eigenvalues are $k + 1, 1 - k$ and $k + 1 > 1 - k$, so in this case $\text{cond}_2(A) = \dfrac{k+1}{1-k}$
    If $k = 0$, then the eigenvalues are 1 and -1, so in this case $\text{cond}_2(A) = 1$
    If $-1 < k < 0$, then the absolute values of the eigenvalues are $k+1, 1-k$ and $1-k > k+1$, so in this case $\text{cond}_2(A) = \dfrac{1-k}{k+1}$
    If $k < -1$, then the absolute values of the eigenvalues are $-k - 1, 1 - k$ and $1 - k > -1 - k$, so in this case $\text{cond}_2(A) = \dfrac{1-k}{-1-k}$